

THE FEFFERMAN-STEIN TYPE INEQUALITIES FOR STRONG AND DIRECTIONAL MAXIMAL OPERATORS IN THE PLANE

HIROKI SAITO AND HITOSHI TANAKA

ABSTRACT. The Fefferman-Stein type inequalities for strong maximal operator and directional maximal operator are verified with composition of the Hardy-Littlewood maximal operator in the plane.

1. INTRODUCTION

The purpose of this paper is to develop a theory of weights for strong maximal operator and directional maximal operator in the plane. We first fix some notations. By weights we will always mean non-negative and locally integrable functions on \mathbb{R}^n . Given a measurable set E and a weight w , $w(E) = \int_E w(x) dx$, $|E|$ denotes the Lebesgue measure of E and 1_E denotes the characteristic function of E . Let $0 < p \leq \infty$ and w be a weight. We define the weighted Lebesgue space $L^p(\mathbb{R}^n, w)$ to be a Banach space equipped with the norm (or quasi norm)

$$\|f\|_{L^p(\mathbb{R}^n, w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p}.$$

For a locally integrable function f on \mathbb{R}^n , we define the Hardy-Littlewood maximal operator \mathfrak{M}_Q by

$$\mathfrak{M}_Q f(x) = \sup_{Q \in \mathcal{Q}} 1_Q(x) \int_Q |f(y)| dy,$$

where \mathcal{Q} is the set of all cubes in \mathbb{R}^n (with sides not necessarily parallel to the axes) and the barred integral $\int_Q |f(y)| dy$ stands for the usual integral average of f over Q . For a locally integrable function f on \mathbb{R}^n , we define the strong maximal operator $\mathfrak{M}_{\mathcal{R}}$ by

$$\mathfrak{M}_{\mathcal{R}} f(x) = \sup_{R \in \mathcal{R}} 1_R(x) \int_R |f(y)| dy,$$

where \mathcal{R} is the set of all rectangles in \mathbb{R}^n with sides parallel to the coordinate axes.

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Let $\mathfrak{T} : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$, $p > 1$, be a sublinear operator. It is a fundamental problem of the weight theory that to determine some maximal operator $\mathfrak{M}_{\mathfrak{T}}$ capturing certain geometric characteristics of \mathfrak{T} such that

$$(1.1) \quad \int_{\mathbb{R}^n} |\mathfrak{T}f(x)|^p w(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p \mathfrak{M}_{\mathfrak{T}}w(x) dx$$

holds for arbitrary weight w . It is well known that

$$\int_{\mathbb{R}^n} \mathfrak{M}_{\mathcal{Q}}f(x)^p w(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p \mathfrak{M}_{\mathcal{Q}}w(x) dx$$

holds for arbitrary weight w and $p > 1$, and further that

$$(1.2) \quad \sup_{t>0} t w(\{x \in \mathbb{R}^n : \mathfrak{M}_{\mathcal{Q}}f(x) > t\}) \leq C \int_{\mathbb{R}^n} |f(x)| \mathfrak{M}_{\mathcal{Q}}w(x) dx.$$

These are called the Fefferman-Stein inequality and are toy models of the problem (1.1) (see [3]).

There is a problem in the book [4, p472]:

Problem 1.1. *Does the analogue of the Fefferman-Stein inequality hold for the strong maximal operator, i.e.*

$$(1.3) \quad \int_{\mathbb{R}^n} \mathfrak{M}_{\mathcal{R}}f(x)^p w(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p \mathfrak{M}_{\mathcal{R}}w(x) dx, \quad p > 1,$$

for arbitrary $w(x) \geq 0$?

Concerning Problem (1.1) it is known that, see [8] (also [11, 12]), (1.3) holds for all $p > 1$ if $w \in A_{\infty}^*$.

We say that w belongs to the class A_p^* whenever

$$[w]_{A_p^*} = \sup_{R \in \mathcal{R}} \int_R w(x) dx \left(\int_R w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty, \quad 1 < p < \infty,$$

$$[w]_{A_1^*} = \sup_{R \in \mathcal{R}} \frac{\int_R w(x) dx}{\operatorname{ess\,inf}_{x \in R} w(x)} < \infty.$$

It follows by Hölder's inequality that the A_p^* classes are increasing, that is, for $1 \leq p \leq q < \infty$ we have $A_p^* \subset A_q^*$. Thus one defines

$$A_{\infty}^* = \bigcup_{p>1} A_p^*.$$

The endpoint behavior of $\mathfrak{M}_{\mathcal{R}}$ close to L^1 is given by Mitsis [10] (for $n = 2$) and Luque and Parissis [9] (for $n > 2$). That is,

$$w(\{x \in \mathbb{R}^n : \mathfrak{M}_{\mathcal{R}}f(x) > t\})$$

$$\leq C \int_{\mathbb{R}^n} \frac{|f(x)|}{t} \left(1 + \left(\log^+ \frac{|f(x)|}{t} \right)^{n-1} \right) \mathfrak{M}_{\mathcal{R}}w(x) dx, \quad t > 0,$$

holds for any $w \in A_{\infty}^*$, where $\log^+ t = \max(0, \log t)$.

In this paper concerning Problem (1.1) we shall establish the following.

Theorem 1.2. *Let w be any weight on \mathbb{R}^2 and set $W = \mathfrak{M}_{\mathcal{R}}\mathfrak{M}_{\mathcal{Q}}w$. Then*

$$\begin{aligned} & w(\{x \in \mathbb{R}^2 : \mathfrak{M}_{\mathcal{R}}f(x) > t\}) \\ & \leq C \int_{\mathbb{R}^2} \frac{|f(x)|}{t} \left(1 + \log^+ \frac{|f(x)|}{t}\right) W(x) dx, \quad t > 0, \end{aligned}$$

holds, where the constant $C > 0$ does not depend on w and f .

By interpolation, we have the following corollary.

Corollary 1.3. *Let w be any weight on \mathbb{R}^2 and set $W = \mathfrak{M}_{\mathcal{R}}\mathfrak{M}_{\mathcal{Q}}w$. Then, for $p > 1$, there exists a constant $C_p > 0$ such that*

$$\|\mathfrak{M}_{\mathcal{R}}f\|_{L^p(\mathbb{R}^2, w)} \leq C_p \|f\|_{L^p(\mathbb{R}^2, W)}$$

holds for all $f \in L^p(\mathbb{R}^2, W)$.

Let Σ be a set of unit vectors in \mathbb{R}^2 , i.e., a subset of the unit circle S^1 . Associated with Σ , we define \mathcal{B}_{Σ} to be the set of all rectangles in \mathbb{R}^2 whose longest side is parallel to some vector in Σ . For a locally integrable function f on \mathbb{R}^2 , we also define the directional maximal operator \mathfrak{M}_{Σ} associated with Σ as

$$\mathfrak{M}_{\Sigma}f(x) = \sup_{R \in \mathcal{B}_{\Sigma}} 1_R(x) \int_R |f(y)| dy.$$

Many authors studied this operator, see [1, 2, 6, 7, 13, 14], and Katz showed that \mathfrak{M}_{Σ} is bounded on $L^2(\mathbb{R}^2)$ with the operator norm $O(\log N)$ for any set Σ with cardinality N .

For fixed sufficiently large integer N , let

$$\Sigma_N = \left\{ \left(\cos \frac{2\pi j}{N}, \sin \frac{2\pi j}{N} \right) : j = 0, 1, \dots, N-1 \right\}$$

be the set of N uniformly spread directions on the circle S^1 . In this paper we shall prove the following, which is a weighted version of the classical result due to Strömberg [14].

Theorem 1.4. *Let $N > 10$ and w be any weight on \mathbb{R}^2 . Set $W = \mathfrak{M}_{\Sigma_N}\mathfrak{M}_{\mathcal{Q}}w$. Then*

$$(1.4) \quad \sup_{t>0} t w(\{x \in \mathbb{R}^2 : \mathfrak{M}_{\Sigma_N}f(x) > t\})^{1/2} \leq C(\log N)^{1/2} \|f\|_{L^2(\mathbb{R}^2, W)}$$

holds for all $f \in L^2(\mathbb{R}^2, W)$, where the constant $C > 0$ does not depend on w and f .

By interpolation, we have the following corollary.

Corollary 1.5. *Let $N > 10$ and w be any weight on \mathbb{R}^2 . Set $W = \mathfrak{M}_{\Sigma_N}\mathfrak{M}_{\mathcal{Q}}w$. Then, for $2 < p < \infty$, there exists a constant $C_p > 0$ such that*

$$\|\mathfrak{M}_{\Sigma_N}f\|_{L^p(\mathbb{R}^2, w)} \leq C_p (\log N)^{1/p} \|f\|_{L^p(\mathbb{R}^2, W)}$$

holds for all $f \in L^p(\mathbb{R}^2, W)$.

The letter C will be used for the positive finite constants that may change from one occurrence to another. Constants with subscripts, such as C_1 , C_2 , do not change in different occurrences.

2. PROOF OF THEOREM 1.2

In what follows we shall prove Theorem 1.2. Our proof relies upon the refinement of the arguments in [10]. With a standard argument, we may assume that the basis \mathcal{R} is the set of all dyadic rectangles R (cartesian products of dyadic intervals) having long side pointing in the x_1 -direction. We denote by P_i , $i = 1, 2$, the projection on the x_i -axis. Fix $t > 0$ and given the finite collection of dyadic rectangles $\{R_i\}_{i=1}^M \subset \mathcal{R}$ such that

$$(2.1) \quad \int_{R_i} |f(y)| dy > t, \quad i = 1, 2, \dots, M.$$

It suffices to estimate $w\left(\bigcup_{i=1}^M R_i\right)$ (see the next section for details).

First relabel if necessary so that the R_i are ordered so that their long sidelengths $|P_1(R_i)|$ decrease. We now give a selection procedure to find subcollection $\{\tilde{R}_i\}_{i=1}^N \subset \{R_i\}_{i=1}^M$.

Take $\tilde{R}_1 = R_1$ and let \tilde{R}_2 be the first rectangle R_j such that

$$|R_j \cap \tilde{R}_1| < \frac{1}{3}|R_j|.$$

Suppose have now chosen the rectangles $\tilde{R}_1, \tilde{R}_2, \dots, \tilde{R}_{i-1}$. We select \tilde{R}_i to be the first rectangle R_j occurring after \tilde{R}_{i-1} so that

$$\left| \bigcup_{k=1}^{i-1} R_j \cap \tilde{R}_k \right| < \frac{1}{3}|R_j|.$$

Thus, we see that

$$(2.2) \quad \left| \bigcup_{j=1}^{i-1} \tilde{R}_i \cap \tilde{R}_j \right| < \frac{1}{3}|\tilde{R}_i|, \quad i = 2, 3, \dots, N.$$

We claim that

$$(2.3) \quad \bigcup_{i=1}^M R_i \subset \left\{ x \in \mathbb{R}^2 : \mathfrak{M}_{\mathcal{Q}}[1_{\bigcup_{i=1}^N \tilde{R}_i}](x) \geq \frac{1}{3} \right\}.$$

Indeed, choose any point x inside a rectangle R_j that is not one of the selected rectangles \tilde{R}_i . Then, there exists a unique $K \leq N$ such that

$$\left| \bigcup_{i=1}^K R_j \cap \tilde{R}_i \right| \geq \frac{1}{3}|R_j|.$$

Since, $|P_1(R_j)| \leq |P_1(\tilde{R}_i)|$ for $i = 1, 2, \dots, K$, we have

$$P_1(R_j) \cap P_1(\tilde{R}_i) = P_1(R_j) \text{ when } R_j \cap \tilde{R}_i \neq \emptyset,$$

where we have used the dyadic structure;

$$(2.4) \quad \text{If both } I \text{ and } J \text{ are the dyadic interval then } I \cap J \in \{I, J, \emptyset\}.$$

Thus,

$$\bigcup_{i=1}^K R_j \cap \tilde{R}_i = \bigcup_{i=1}^K P_1(R_j) \times (P_2(R_j) \cap P_2(\tilde{R}_i)) = P_1(R_j) \times \bigcup_{i=1}^K P_2(R_j) \cap P_2(\tilde{R}_i).$$

Hence,

$$\left| \bigcup_{i=1}^K P_2(R_j) \cap P_2(\tilde{R}_i) \right| \geq \frac{1}{3} |P_2(R_j)|.$$

Thanks to the fact that $|P_2(R_j)| \leq |P_1(R_j)| \leq |P_1(\tilde{R}_i)|$, this implies that

$$\left| \bigcup_{i=1}^K Q \cap \tilde{R}_i \right| \geq \frac{1}{3} |Q|,$$

where Q is a unique dyadic cube containing x and having the side length $|P_2(R_j)|$. This proves (2.3).

It follows from (2.3) and (1.2) that

$$\begin{aligned} w \left(\bigcup_{i=1}^M R_i \right) &\leq w \left(\left\{ x \in \mathbb{R}^2 : \mathfrak{M}_{\mathcal{Q}}[1_{\bigcup_{i=1}^N \tilde{R}_i}](x) \geq \frac{1}{3} \right\} \right) \\ &\leq CU \left(\bigcup_{i=1}^N \tilde{R}_i \right) \leq C \sum_{i=1}^N U(\tilde{R}_i), \end{aligned}$$

where $U = \mathfrak{M}_{\mathcal{Q}} w$. We shall evaluate the quantity

$$(i) = \sum_{i=1}^N U(\tilde{R}_i).$$

Let $\mu_U(x)$ be the weighted multiplicity function associated to the family $\{\tilde{R}_i\}$, that is,

$$\mu_U(x) = \sum_{i=1}^N \frac{U(\tilde{R}_i)}{|\tilde{R}_i|} 1_{\tilde{R}_i}(x).$$

By (2.1), choosing δ_0 small enough determined later,

$$\begin{aligned} (i) &\leq \sum_{i=1}^N \frac{U(\tilde{R}_i)}{|\tilde{R}_i|} \int_{\tilde{R}_i} \frac{|f(y)|}{t} dy \\ &= \delta_0 \int_{\mathbb{R}^2} \mu_U(x) W(x)^{-1} \cdot \frac{|f(x)|}{\delta_0 t} \cdot W(x) dx. \end{aligned}$$

Using the elementary inequality

$$ab \leq (e^a - 1) + b(1 + \log^+ b), \quad a, b \geq 0,$$

we get

$$\begin{aligned}
\text{(i)} &\leq \delta_0 \int_{\mathbb{R}^2} (\exp(\mu_U(x)W(x)^{-1}) - 1) W(x) dx \\
&\quad + \delta_0 \int_{\mathbb{R}^2} \frac{|f(x)|}{\delta_0 t} \left(1 + \log^+ \frac{|f(x)|}{\delta_0 t}\right) W(x) dx \\
&\leq \delta_0 \int_{\mathbb{R}^2} (\exp(\mu_U(x)W(x)^{-1}) - 1) W(x) dx \\
&\quad + (1 - \log \delta_0) \int_{\mathbb{R}^2} \frac{|f(x)|}{t} \left(1 + \log^+ \frac{|f(x)|}{t}\right) W(x) dx.
\end{aligned}$$

We have to evaluate the quantity

$$\text{(ii)} = \int_{\mathbb{R}^2} (\exp(\mu_U(x)W(x)^{-1}) - 1) W(x) dx.$$

We expand the exponential in a Taylor series. Then

$$\begin{aligned}
\text{(ii)} &= \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{R}^2} (\mu_U(x)W(x)^{-1})^k W(x) dx \\
&= \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{R}^2} \mu_U(x)^k W(x)^{1-k} dx.
\end{aligned}$$

Fix $k \geq 2$. We use an elementary inequality

$$\left(\sum_{i=1}^{\infty} a_i\right)^k \leq k \sum_{i=1}^{\infty} a_i \left(\sum_{j=1}^i a_j\right)^{k-1},$$

where $\{a_i\}_{i=1}^{\infty}$ is a sequence of summable nonnegative reals. Then

$$\begin{aligned}
&\int_{\mathbb{R}^2} \mu_U(x)^k W(x)^{1-k} dx \\
&\leq k \sum_{i=1}^N \frac{U(\tilde{R}_i)}{|\tilde{R}_i|} \int_{\tilde{R}_i} \left(\sum_{j=1}^i \frac{U(\tilde{R}_j)}{|\tilde{R}_j|} 1_{\tilde{R}_j}(x)\right)^{k-1} W(x)^{1-k} dx \\
&\leq k \sum_{i=1}^N \frac{U(\tilde{R}_i)}{|\tilde{R}_i|} \int_{\tilde{R}_i} \left(\sum_{j=1}^i 1_{\tilde{R}_j}(x)\right)^{k-1} dx,
\end{aligned}$$

where we have used

$$\sum_{j=1}^i \frac{U(\tilde{R}_j)}{|\tilde{R}_j|} 1_{\tilde{R}_j}(x) \leq \left(\sum_{j=1}^i 1_{\tilde{R}_j}(x)\right) W(x).$$

We claim that, for $n = 1, 2, \dots, N$,

$$(2.5) \quad |X_{i,n}| \leq 3^{1-n} |\tilde{R}_i|,$$

where

$$X_{i,n} = \left\{x \in \tilde{R}_i : \sum_{j=1}^i 1_{\tilde{R}_j}(x) \geq n\right\}.$$

Indeed, first we notice that, for any k and j with $N \geq k > j \geq 1$, if $\tilde{R}_k \cap \tilde{R}_j \neq \emptyset$, then

$$\tilde{R}_k \cap \tilde{R}_j = P_1(\tilde{R}_k) \times P_2(\tilde{R}_j),$$

because we have $P_1(\tilde{R}_k) \subset P_1(\tilde{R}_j)$ and, by (2.2), $|P_2(\tilde{R}_k) \cap P_2(\tilde{R}_j)| < \frac{1}{3}|P_2(\tilde{R}_k)|$. With this in mind, we can observe the following:

There exists a set of dyadic intervals $\{I_{jk}\}$ with $j = 1, 2, \dots, n$ and $k = 1, 2, \dots, K_j$ that satisfies the following:

- The dyadic intervals I_{jk} are pairwise disjoint for varying k ;
- For each I_{jk} , $j > 1$, there exists a unique $I_{(j-1)l} \supsetneq I_{jk}$;
- For each I_{jk} there exists a unique number $i_{jk} \leq i$ such that $I_{jk} = P_2(\tilde{R}_{i_{jk}})$;
- $P_2(X_{i,1}) = I_{11}$, $P_2(X_{i,2}) = \bigcup_{k=1}^{K_2} I_{2k}$, \dots , $P_2(X_{i,j}) = \bigcup_{k=1}^{K_j} I_{jk}$, \dots , $P_2(X_{i,n}) = \bigcup_{k=1}^{K_n} I_{nk}$;
- If $I_{jk} \subset I_{(j-1)l}$, then $i_{jk} < i_{(j-1)l}$ and $\tilde{R}_{i_{(j-1)l}} \cap \tilde{R}_{i_{jk}} \neq \emptyset$.

It follows from the last relation and (2.2) that

$$3 \sum_{k=1}^{K_j} |I_{jk}| < \sum_{k=1}^{K_{j-1}} |I_{(j-1)k}|, \quad j = 2, 3, \dots, n.$$

This gives us that

$$3^{n-1} \sum_{k=1}^{K_n} |I_{nk}| < |I_{11}|,$$

which yields (2.5).

It follows from (2.5) that

$$\begin{aligned} & \frac{U(\tilde{R}_i)}{|\tilde{R}_i|} \int_{\tilde{R}_i} \left(\sum_{j=1}^i 1_{\tilde{R}_j}(x) \right)^{k-1} dx \\ & \leq \frac{U(\tilde{R}_i)}{|\tilde{R}_i|} \sum_{n=1}^N n^{k-1} |X_{i,n}| \\ & \leq U(\tilde{R}_i) \sum_{n=1}^N n^{k-1} 3^{1-n}. \end{aligned}$$

Altogether, the quantity (ii) can be majorized by

$$(i) \times \left[1 + \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \sum_{n=1}^N n^{k-1} 3^{1-n} \right].$$

There holds

$$\sum_{k=2}^{\infty} \frac{1}{(k-1)!} \sum_{n=1}^N n^{k-1} 3^{1-n} \leq 3 \sum_{n=1}^{\infty} \left(\frac{e}{3} \right)^n =: C_0.$$

If we choose δ_0 so that $\delta_0(1 + C_0) = \frac{1}{2}$, we obtain

$$(i) \leq C \int_{\mathbb{R}^2} \frac{|f(x)|}{t} \left(1 + \log^+ \frac{|f(x)|}{t} \right) W(x) dx.$$

This completes the proof.

Remark. Since our proof relies only upon the dyadic structure (2.4), it can be applied the basis \mathcal{R} of the form the set of all rectangles in \mathbb{R}^n whose sides parallel to the coordinate axes and which are congruent to the rectangle $(0, a)^{n-1} \times (0, b)$ with varying $a, b > 0$.

3. PROOF OF THEOREM 1.4

In what follows we shall prove Theorem 1.4. We follow the argument in [5] Chapter 10, Theorem 10.3.5. To avoid problems with antipodal points, it is convenient to split Σ_N as the union of eight sets, in each of which the angle between any two vectors does not exceed $\pi/4$. It suffices therefore to obtain (1.4) for each such subset of Σ_N . Let us fix one such subset of Σ_N , which we call Σ_N^1 .

To prove (1.4), we fix a $t > 0$ and we start with a compact subset K of the set $\{x \in \mathbb{R}^2 : \mathfrak{M}_{\Sigma_N^1} f(x) > t\}$. Then for every $x \in K$, there exists an open rectangle R_x that contains x and whose longest side is parallel to a vector in Σ_N^1 . By compactness of K , there exists a finite subfamily $\{R_\alpha\}_{\alpha \in \mathcal{A}}$ of the family $\{R_x\}_{x \in K}$ such that

$$(3.1) \quad \int_{R_\alpha} |f(y)| dy > t$$

for all $\alpha \in \mathcal{A}$ and such that the union of the R_α 's covers K .

In the sequel we denote by θ_α the angle between the x_1 -axis and the vector pointing in the longer direction of R_α for any $\alpha \in \mathcal{A}$. We also denote by l_α the shorter side of R_α and by L_α the longer side of R_α for any $\alpha \in \mathcal{A}$.

We shall select the subfamily $\{R_\beta\}_{\beta \in \mathcal{B}}$ as follows:

Without loss of generality we may assume that $\mathcal{A} = \{1, 2, \dots, \ell\}$ with $L_j \geq L_{j+1}$ for all $j = 1, 2, \dots, \ell - 1$. Let $\beta_1 = 1$ and choose β_2 to be the first number in $\{\beta_1 + 1, \beta_1 + 2, \dots, \ell\}$ such that

$$|R_{\beta_1} \cap R_{\beta_2}| \leq \frac{1}{2} |R_{\beta_2}|.$$

We next choose β_3 to be the first number in $\{\beta_2 + 1, \beta_2 + 2, \dots, \ell\}$ such that

$$|R_{\beta_1} \cap R_{\beta_3}| + |R_{\beta_2} \cap R_{\beta_3}| \leq \frac{1}{2} |R_{\beta_3}|.$$

Suppose we have chosen the numbers $\beta_1, \beta_2, \dots, \beta_{j-1}$. Then we choose β_j to be the first number in $\{\beta_{j-1} + 1, \beta_{j-1} + 2, \dots, \ell\}$ such that

$$(3.2) \quad \sum_{k=1}^{j-1} |R_{\beta_k} \cap R_{\beta_j}| \leq \frac{1}{2} |R_{\beta_j}|.$$

Since the set \mathcal{A} is finite, this selection stops after m steps.

Define $\mathcal{B} = \{\beta_1, \beta_2, \dots, \beta_m\}$ and let

$$Y(x) = \sum_{\beta \in \mathcal{B}} 1_{(R_\beta)^*}(x),$$

where $(R_\beta)^*$ is the rectangle R_β expanded 5 times in both directions.

We claim that

$$(3.3) \quad w(K) \leq w\left(\bigcup_{\alpha \in \mathcal{A}} R_\alpha\right) \leq C(\log N) \int_{\mathbb{R}^2} Y(x) U(x) dx,$$

where $U(x) = \mathfrak{M}_Q w(x)$. To verify this claim, we need the following lemma.

We set $\omega_k = \frac{2\pi 2^k}{N}$ for $k \in \mathbb{Z}^+$ and $\omega_0 = 0$. We let $M = \lfloor \frac{\log(N/8)}{\log 2} \rfloor$.

Lemma 3.1 ([5, Lemma 10.3.6]). *Let R_α be a rectangle in the family $\{R_\alpha\}_{\alpha \in \mathcal{A}}$ and let $0 \leq k < M$. Suppose that $\beta \in \mathcal{B}$ is such that*

$$\omega_k \leq |\theta_\alpha - \theta_\beta| < \omega_{k+1}$$

and such that $L_\beta \geq L_\alpha$. Let

$$s_\alpha = 8 \max(l_\alpha, \omega_k L_\alpha).$$

For an arbitrary $x \in R_\alpha$, let Q be a square centered at x with sides of length s_α parallel to the sides of R_α . Then we have

$$(3.4) \quad \frac{|R_\beta \cap R_\alpha|}{|R_\alpha|} \leq 32 \frac{|(R_\beta)^* \cap Q|}{|Q|}.$$

We shall prove (3.3). By (1.2) it suffices to show that

$$(3.5) \quad \bigcup_{\alpha \in \mathcal{A}} R_\alpha \subset \left\{ x \in \mathbb{R}^2 : \mathfrak{M}_Q Y(x) > \frac{C}{\log N} \right\}.$$

Since we may assume that $C/(\log N) < 1$, the set $\bigcup_{\beta \in \mathcal{B}} R_\beta$ is contained in the set of the right hand side of (3.5). So, we fix $\alpha \in \mathcal{A} \setminus \mathcal{B}$. Then the rectangle R_α was not selected in the selection procedure.

By the construction and (3.2), we see that there exists j such that

$$\sum_{k=1}^j |R_{\beta_k} \cap R_\alpha| > \frac{1}{2} |R_\alpha|$$

and such that $L_{\beta_k} \geq L_\alpha$ for all $k = 1, 2, \dots, j$.

Let $\mathcal{B}_j = \{\beta_1, \beta_2, \dots, \beta_j\}$. It follows from Lemma 3.1 that

$$\begin{aligned} \frac{1}{2} &< \sum_{\beta \in \mathcal{B}_j} \frac{|R_\beta \cap R_\alpha|}{|R_\alpha|} \\ &= \sum_{k=0}^M \sum_{\substack{\beta \in \mathcal{B}_j: \\ \omega_k \leq |\theta_\alpha - \theta_\beta| < \omega_{k+1}}} \frac{|R_\beta \cap R_\alpha|}{|R_\alpha|} \\ &\leq 32 \sum_{k=0}^M \sum_{\substack{\beta \in \mathcal{B}_j: \\ \omega_k \leq |\theta_\alpha - \theta_\beta| < \omega_{k+1}}} \frac{|(R_\beta)^* \cap Q_k|}{|Q_k|}, \end{aligned}$$

where Q_k is a square determined by Lemma 3.1 with an arbitrary $x \in R_\alpha$. Since we have $M \leq C(\log N)$ and

$$\sum_{\substack{\beta \in \mathcal{B}_j: \\ \omega_k \leq |\theta_\alpha - \theta_\beta| < \omega_{k+1}}} \frac{|(R_\beta)^* \cap Q_k|}{|Q_k|} \leq C \mathfrak{M}_{\mathcal{Q}} Y(x) \text{ for all } x \in R_\alpha,$$

we obtain

$$\mathfrak{M}_{\mathcal{Q}} Y(x) > \frac{C}{\log N} \text{ for all } x \in R_\alpha,$$

which implies (3.5) and, hence, (3.3).

We now evaluate

$$(i) = \int_{\mathbb{R}^2} Y(x) U(x) dx = \sum_{\beta \in \mathcal{B}} U((R_\beta)^*).$$

By (3.1) and Hölder's inequality we have

$$\begin{aligned} (i) &\leq \frac{1}{t} \sum_{\beta \in \mathcal{B}} U((R_\beta)^*) \int_{R_\beta} |f(y)| dy \\ &= \frac{1}{t} \int_{\mathbb{R}^2} \left(\sum_{\beta \in \mathcal{B}} \frac{U((R_\beta)^*)}{|R_\beta|} 1_{R_\beta}(y) \right) |f(y)| dy \\ &\leq \frac{1}{t} \left(\int_{\mathbb{R}^2} \left(\sum_{\beta \in \mathcal{B}} \frac{U((R_\beta)^*)}{|R_\beta|} 1_{R_\beta}(y) \right)^2 W(y)^{-1} dy \right)^{1/2} \|f\|_{L^2(\mathbb{R}^2, W)}. \end{aligned}$$

We have further

$$\begin{aligned} (ii) &= \int_{\mathbb{R}^2} \left(\sum_{\beta \in \mathcal{B}} \frac{U((R_\beta)^*)}{|R_\beta|} 1_{R_\beta}(y) \right)^2 W(y)^{-1} dy \\ &= \sum_{j=1}^m \left(\frac{U((R_{\beta_j})^*)}{|R_{\beta_j}|} \right)^2 \int_{R_{\beta_j}} W(y)^{-1} dy \\ &\quad + 2 \sum_{j=1}^m \frac{U((R_{\beta_j})^*)}{|R_{\beta_j}|} \sum_{k=1}^{j-1} \frac{U((R_{\beta_k})^*)}{|R_{\beta_k}|} \int_{R_{\beta_k} \cap R_{\beta_j}} W(y)^{-1} dy. \end{aligned}$$

We notice that, for any $y \in R_{\beta_k} \cap R_{\beta_j}$

$$W(y) \geq \frac{U((R_{\beta_k})^*)}{|(R_{\beta_k})^*|} = \frac{U((R_{\beta_k})^*)}{25|R_{\beta_k}|}.$$

This yields

$$\begin{aligned}
\text{(ii)} &\leq 25 \sum_{j=1}^m U((R_{\beta_j})^*) \\
&\quad + 50 \sum_{j=1}^m \frac{U((R_{\beta_j})^*)}{|R_{\beta_j}|} \sum_{k=1}^{j-1} |R_{\beta_k} \cap R_{\beta_j}| \\
&\leq 50 \sum_{\beta \in \mathcal{B}} U((R_{\beta_j})^*),
\end{aligned}$$

where we have used (3.2). Altogether, we obtain

$$\text{(i)} \leq \frac{C}{t^2} \|f\|_{L^2(\mathbb{R}^2, W)}^2,$$

which yields by (3.3)

$$w(K) \leq \frac{C(\log N)}{t^2} \|f\|_{L^2(\mathbb{R}^2, W)}^2.$$

Since K was an arbitrary compact subset of $\{x \in \mathbb{R}^2 : \mathfrak{M}_{\Sigma_N^1} f(x) > t\}$, the same estimate is valid for the latter set and we finish the proof.

REFERENCES

- [1] A. Alfonseca, F. Soria and A. Vargas, *A remark on maximal operators along directions in \mathbb{R}^2* , Math. Res. Lett., **10** (2003), no. 1, 41–49.
- [2] ———, *An almost-orthogonality principle in L^2 for directional maximal functions*, Harmonic analysis at Mount Holyoke (South Hadley, MA, 2001), 1–7, Contemp. Math., **320**, Amer. Math. Soc., Providence, RI, 2003.
- [3] C. Fefferman and E. M. Stein *Some Maximal Inequalities*, Amer. J. math., **93** (1971), no. 1, 107–115.
- [4] J. Garcia-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, Math. Stud., **116** (1985).
- [5] L. Grafakos, *Modern Fourier Analysis*, volume 250 of Graduate Texts in Mathematics, Springer, New York, 2nd edition, 2008.
- [6] N. H. Katz, *Maximal operators over arbitrary sets of directions*, Duke Math. J., **97** (1999), no. 1, 67–79.
- [7] ———, *Remarks on maximal operators over arbitrary sets of directions*, Bull. London Math. Soc., **31** (1999), no. 6, 700–710.
- [8] Kai-Ching Lin, Ph.D. University of California, Los Angeles 1984 United States. Dissertation: Harmonic Analysis on the Bidisc.
- [9] T. Luque and I. Parissis *The endpoint Fefferman-Stein inequality for the strong maximal function*, J. Funct. Anal. **266** (2014), no. 1, 199–212.
- [10] T. Mitsis, *The weighted weak type inequality for the strong maximal function*, J. Fourier Anal. Appl. **12** (2006), no. 6, 645–652.
- [11] C. Pérez, *Weighted norm inequalities for general maximal operators*, Conference on Mathematical Analysis (El Escorial, 1989), Publ. Mat. **35** (1991), no. 1, 169–186.
- [12] ———, *A remark on weighted inequalities for general maximal operators*, Proc. Amer. Math. Soc., **119** (1993), no. 4, 1121–1126.
- [13] H Saito and H. Tanaka, *Directional maximal operators and radial weights on the plane*, Bull. Austral. Math. Soc., **89** (2014), no 2, 397–414.
- [14] J. O. Strömberg, *Maximal functions associated to rectangles with uniformly distributed directions*, Ann. Math. (2), **107** (1978), no. 2, 399–402.

ACADEMIC SUPPORT CENTER, KOGAKUIN UNIVERSITY, 2665-1, NAKANOMACHI,
HACHIOJI-SHI TOKYO, 192-0015, JAPAN

E-mail address: j1107703@gmail.com

RESEARCH AND SUPPORT CENTER ON HIGHER EDUCATION FOR THE HEARING AND
VISUALLY IMPAIRED, NATIONAL UNIVERSITY CORPORATION TSUKUBA UNIVERSITY
OF TECHNOLOGY, KASUGA 4-12-7, TSUKUBA CITY, IBARAKI, 305-8521 JAPAN

E-mail address: htanaka@k.tsukuba-tech.ac.jp